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On the Geometrization of Forces

ANDREAS QUALE†

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

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Abstract

The problem of giving a geometric description of external forces acting on general charged test particles is discussed, within the framework of General Relativity. The concept of a pointlike test particle is analyzed; and we find that it is possible to introduce a strongly localized interaction between the particle and external fields, while still retaining the notion of a test (i.e. non-radiating) particle, in such a way that the particle trajectory is *geodesic* in the local space-time geometry, irrespective of the charges coupling it to external matter fields or to space-time curvature. The phenomenon of *deviation*, due to differential forces between (identically charged) noninteracting point particles, is also considered; we find that a geometric description of such forces requires introducing additional geometric objects into space-time, making it non-Riemannian. It is shown that a quite general differential force may be described geometrically as a *geodesic deviation* in a nonsymmetric affine space, endowed with a *torsion field*. The physical properties of the torsion, and its relation to geometrodynamics, are discussed.

1. *Introduction*: *The Geometrization of Charge*

The central idea of geometrodynamics, or 'already unified field theory' (Wheeler, 1962), in General Relativity may be briefly outlined as follows: Let ϕ denote a set of matter (i.e. nongravitational) field, with field equations derivable from an action principle‡

$$\frac{\delta L_M}{\delta\phi} = 0, \qquad L_M = L_M(\phi; \phi_{,k}; \ldots) \tag{1.1}$$

of some specified differential order. The symmetrized matter energy tensor $E_{ik}(\phi; \phi_{,l}; \ldots)$, as derived from (1.1) in the usual way, will then in general satisfy a number of structural relations—i.e. algebraic conditions on its components, characteristic of the fields ϕ—which follow from the form of

† On leave of absence 1970–71 from: Department of Physics, Oslo University, Oslo, Norway.

‡ Our notation is as follows: Lower case Latin (Greek) indices range and sum over 1,...4 (1,...3); the metric has the signature of the Lorentz matrix $\eta_{ik} = \mathrm{diag}(1,1,1,-1)$; commas denote partial derivatives; the signs of the Riemann and Ricci tensors are determined by the definitions

$$R^i{}_{klm} \equiv 2[\Gamma^i{}_{k[l,m]} + \Gamma^s{}_{k[l}\,\Gamma^i{}_{m]s}$$
$$R_{ik} \equiv R^s{}_{iks}.$$

the Lagrangian (1.1). Moreover, E_{ik} is the source of a gravitational field, namely the metric tensor g_{ik}, according to Einstein's field equations

$$\frac{1}{\sqrt{(-g)}} \cdot \frac{\delta}{\delta g_{ik}} (\sqrt{(-g)}\, R + L_M) = G^{ik} - E^{ik} = 0 \tag{1.2}$$

in suitable units. Regarding (1.1)–(1.2) as a set of coupled equations in ϕ and g_{ik}, we may ask whether this set is restrictive enough to yield a solution, i.e. a local or functional relation

$$\phi = \phi(g_{ik}; g_{ik,l}; \ldots) \tag{1.3}$$

for ϕ in terms of the metric tensor and its derivatives. If such a solution can be obtained in a physically unique way [i.e. up to any arbitrariness in gauge, phase, etc., inherent in the original physical description (1.1) of ϕ], then the fields ϕ are said to be *geometrized.* This means that the physical laws governing ϕ may be written as laws governing space-time geometry. Equations (1.1) thus, by (1.3), takes the form of a set of equations governing the *metric tensor.* In other words: the physics of the fields ϕ may be described completely as the behaviour of a certain type of gravitational field; the field equations (1.1), when (1.3) is substituted, then *restrict* the geometry of space-time in such a way that it exhibits just those observable phenomena usually attributed to the presence of the matter fields.

The problem is, of course, to obtain formally the solution (1.3). This has been done in a number of important special cases, such as the electromagnetic field (Wheeler, 1962, see in particular, pp. 16–87 and 225–253), scalar meson fields (Kuchař, 1963; Penney, 1965), and also in part for the Weyl spinor neutrino field (Bergmann, 1960). In each case the field equations (1.1) were explicitly employed; and a general proof of the feasibility of such a geometrization of matter fields would presumably require a suitable delimitation of the various types of interactions and couplings allowed in the theory.

In this paper we shall discuss the question of whether it is possible (or even meaningful) to perform a similar geometrization of the action of a field on charged particles; more specifically we consider the motion of a *test particle* in some external field θ. ($\theta \equiv \{\phi, g_{ik}\}$ here denotes a set of unspecified matter fields ϕ *plus* the gravitational field g_{ik}). We define a test particle as a timelike trajectory, on which the fields θ are nonsingular, corresponding to the limiting procedure of Infeld & Schild (1949) and Chase (1954), who consider neutral and electrically charged mass monopoles, respectively. However, we allow our particle to carry any number of scalar characteristics, such as mass and various kinds of charge (electric, mesonic, etc.); moreover, it may possess directive properties, such as multipole structure or intrinsic angular momentum (spin). All these intrinsic attributes, *except mass*, through which the particle is coupled to the fields θ, will be referred to collectively as *dynamical charges.* Our treatment of both fields and particles will be purely classical, and only particles with nonzero rest masses will be considered.

The action of the fields θ on the particle is described by a *force law*:

$$\frac{DU^k}{Ds} = f^k(\phi, g_{ik}, U^k, m, DC) \tag{1.4}$$

where the left-hand side is the absolute derivative (Synge & Schild, 1948) of the unit tangent vector U^k along the trajectory, DC denotes the set of dynamical charges carried by the particle, and the right-hand side is a vector which we shall call the *4-force*. We shall assume that f^k depends on the fields θ and their derivatives up to some finite order on the trajectory, but not on the field values elsewhere, and that it depends *homogeneously* on each DC: in other words, the motion (1.4) is assumed to be geodesic when the particle carries no charge. On the other hand, f^k need not depend homogeneously on ϕ; it is, for instance, well known that a spinning mass dipole will deviate from geodesic motion, due to the coupling of its spin to the Riemann tensor (Papapetrou, 1951).

How can one define a geometrization of the force (1.4) in a physically meaningful way? To see this, we have to look closer at the notion of a test particle. A basic conceptual problem of the theory of point particles is that of their infinite self-fields (which essentially result from the formal requirement that a finite extensive quantity, such as mass, be squeezed into an infinitesimal volume), and in particular the disentangling of these fields from the external or 'background' field acting on the particles. The Infeld–Schild limiting procedure copes with this problem, and in effect defines test particles, by assuming that it is possible to define a background (matter and gravitation) field, as that obtaining in the *absence* of the particles, and that *their presence does not affect this background.* This, which clearly excludes any radiative reaction from a particle, is achieved formally by letting its mass and charges 'go to zero'. [In fact, the very concept of a 'point particle' can only be defined relative to some background field; the particle properties (mass and charges) are thought of as concentrated into a spatial volume so small that the field may be regarded as uniform within it. Thus, for instance, the planets may be treated as point particles in the solar gravitational force field, to the extent that the latter may be considered as uniform over distances of the order of a planetary diameter.] Any observed anomaly of its motion (i.e. deviation from geodicity) is then ascribed to some intrinsic particle attribute (charge) which makes it 'respond' to the action of background matter fields, or to inhomogeneities of the background gravitational field, as determined by the force law (1.4).

However, any particle with nonzero mass and charges, however small, will obviously affect its field environment *at sufficiently small distances*; in other words, the concept of a test particle breaks down if one approaches the particle too closely. Hence, the basic physical assumptions underlying the Infeld–Schild approach may be put as follows: No radiative reaction from the particle is detectable at reasonable distances—i.e. outside what might be called its 'near zone'. The background field observed at these distances is then extrapolated in toward the trajectory 'as if the particle were not there', so to speak: i.e. its value in the near zone is assumed to be that which would have obtained in the absence of the particle. The essential point is that there will always be a small spatial region surrounding the particle (the near zone) where the background field is physically *unobservable*, as long as the particle is treated as a test particle; in particular, the background field values on the trajectory is always deduced by extrapolation in from the far zone. In fact, the only thing we can be reasonably sure of is that the actual field in the near zone, if it were measured, will *not* be equal to the extrapolated background field, since—as noted—the particle will in general affect the field in its immediate vicinity.

The usual notion of a charged test particle thus ignores the effect of the particle on its field environment, and explains its nongeodesic motion in the background geometry $\mathscr{G}$ (as extrapolated in from the far zone) by postulating that the particle carries dynamical charges, coupling it to the external background field θ. We want to consider the hypothesis that a particle *modifies* the space-time geometry in its near zone, in such a way that in this modified geometry $\bar{\mathscr{G}}$ the particle motion is *geodesic*, irrespective of the background field θ on the trajectory. The particle will thus move as a neutral mass monopole in $\bar{\mathscr{G}}$. We shall then say that the force (1.4) on the particle has been *geometrized*: the physical interpretation is that the charges, previously assumed to be internal particle degrees of freedom, are now to be regarded as degrees of freedom in the *local* space-time geometry $\bar{\mathscr{G}}$ on the trajectory.

Summarizing briefly: the usual conception of a test particle is that it does not affect the field in its vicinity; the field-particle coupling is one-way, and mediated by the charges. The external forces then appear as *fixed constraints*, determining the particle motion. In reality, however, there will of course always be an *interaction* between particle and field: a test particle is, in effect, an idealized description of the physical situation when the effects of this interaction are imperceptible outside a small spatial region $\mathscr{T}$ (the near zone) surrounding the particle. We are investigating the possibility that this localized interaction may modify the space-time geometry $\mathscr{G}$ in $\mathscr{T}$, in such a way that the modified geometry $\bar{\mathscr{G}}$ can take over the functions of the charges coupling the particle to θ: i.e. that it will constrain the particle to move *as if uncharged*, irrespective of the external fields θ on its trajectory. At the same time, we want to retain the notion of a *test particle*, acted on by an external field, but not reacting back upon it; hence we shall still treat the modified geometry $\bar{\mathscr{G}}$ as a fixed constraint on the particle motion.

In the following sections, we show (Section 2) that it is possible to geometrize, in the sense defined above, *any 4-force*—i.e. the action of any set of external fields on a particle carrying any combination of dynamical charges—by suitably modifying the near zone geometry. We find that this modification does not affect the validity of the matter and gravitational field equations, though it may introduce new matter field sources in the near zone. However, the differential forces between neighbouring identically charged test particles can not be geometrized in this way, essentially because the notion of noninteracting neighbouring pointlike particles is incompatible with that of a finite near zone. In order to describe differential forces in geometric terms, it is necessary to introduce additional geometric objects into space-time, making the latter non-Riemannian. It is shown (Section 3) that it is possible to achieve a geometric description of quite general differential forces by means of a torsion field (i.e. a nonsymmetric connection). In Section 4, we discuss the physical interpretation of our results, and in particular the relation of the torsion to geometrodynamics: i.e. physics described within the framework of purely Riemannian geometry.

2. *The Metric Perturbation*

Let Σ be any space-time region satisfying the Lichnerowicz regularity criterion [i.e. at least C^3 in the metric, except possibly for junction 3-surfaces, on which it is at least C^1, compare Synge (1960)], and let C be any smooth nongeodesic timelike trajectory in Σ, described parametrically in some coordinate frame (x) covering Σ by functions $x^k = x^k(s)$ satisfying

$$\frac{D^2 x^k}{Ds^2} \equiv \frac{d^2 x^k}{ds^2} + \Gamma^k_{lm} \frac{dx^l}{ds}\frac{dx^m}{ds} = f^k(s) \tag{2.1}$$

where $\Gamma^k_{lm}(x)$ is the Christoffel connection of the metric tensor $g_{ik}(x)$ in Σ, and s is the arc length of C. Henceforth equalities valid on C, but not necessarily elsewhere in Σ, will be denoted by the sign $\overset{C}{=}$.

We now choose the coordinates (x) to be a *rest frame* of C: i.e. such that the worldline of its spatial origin $(0,0,0,x^4)$ coincides with C, while the coordinate time $x^4 \equiv t$ on C coincides with the (absolute value of the) arc length s of C. This implies that

$$\frac{d^2 x^\alpha}{ds^2} \overset{C}{=} \frac{dx^\alpha}{ds} \overset{C}{=} 0, \qquad dx^4 \overset{C}{=} \sqrt{(-ds^2)} \tag{2.2}$$

so that (2.1) reduces to

$$\Gamma^k_{44} \overset{C}{=} -f^k \tag{2.3}$$

The coordinate transformation $(x) \to (x')$ defined by

$$x^k = x^{k'} - \tfrac{1}{2} x^{\alpha'} x^{l'} \Gamma^k_{\alpha l}(x) \tag{2.4}$$

readily yields

$$\begin{aligned} g'_{ik,\beta} &\overset{C}{=} g'_{44,4} \overset{C}{=} g'_{\alpha\beta,4} \overset{C}{=} 0 \\ g'_{ik,\beta 4} &\overset{C}{=} g'_{44,44} \overset{C}{=} g'_{\alpha\beta,44} \overset{C}{=} 0 \end{aligned} \tag{2.5}$$

Since (2.2), (2.3) are not affected by (2.4), (x') is still a rest frame of C; moreover we may orient spatial axes so that

$$g'_{\alpha\beta} \overset{C}{=} \delta_{\alpha\beta}, \qquad g'_{44} \overset{C}{=} -1 \tag{2.6}$$

the latter as a consequence of (2.2).

[Actually, according to a theorem due to Fermi (1922), it is always possible to normalize the metric and its first derivatives on C to the Minkowski values $g'_{ik} \overset{C}{=} \eta_{ik}$, $g'_{ik,l} \overset{C}{=} 0$; however, this cannot be carried out in a *rest frame*, unless C is geodesic. The normalization (2.5) leaves *three* arbitrary nonzero metric derivatives $g'_{4\alpha,4}$ to allow for an arbitrary nonvanishing 4-force, since lowering the index in (2.3) only yields three nontrivial equations

$$\Gamma'_{\alpha,44} \overset{C}{=} g_{4\alpha,4} \overset{C}{=} -f_\alpha \tag{2.7}$$

(Both $\Gamma'_{4,44}$ and f'_4, but not $f^{4'}$, vanish on C in the rest frame (x').) Henceforth we assume the normalization (2.2), (2.3), (2.5), (2.6) of the coordinates on C, which will be referred to as the *C-gauge*, and drop dashes.]

The Riemann tensor in Σ may be written in the form

$$R_{iklm} = \tfrac{1}{2}(g_{il,km} + g_{km,il} - g_{im,kl} - g_{kl,im}) + g_{rs}(\Gamma^r_{il}\Gamma^s_{km} - \Gamma^r_{im}\Gamma^s_{kl}) \tag{2.8}$$

which reduces on C to

$$R_{iklm} \overset{C}{=} \tfrac{1}{2}(g_{il,km} + g_{km,il} - g_{im,kl} - g_{kl,im}) \tag{2.9}$$

Let C be enclosed in a timelike tube $\mathscr{T}$ of finite cross-section. By this, we mean that the intersection of $\mathscr{T}$ with any instantaneous spacelike hypersurface $t = \text{const.}$ is compact, with a finite 3-volume σ bounded by a 2-surface $\partial\sigma$ of finite area; both σ and $\partial\sigma$ may vary with t. The idea is to assume that the interior of $\mathscr{T}$ is the near zone of a particle with trajectory C; the space-time metric $g_{ik}(x)$ in $\mathscr{T}$ then represents the background metric tensor, as extrapolated in from the far zone, see the discussion in Section 1.

Consider now a finite perturbation of the metric tensor in Σ:

$$g_{ik}(x) \to \bar{g}_{ik}(x) = g_{ik}(x) + h_{ik}(x) \tag{2.10}$$

where the perturbation functions h_{ik} have the following form:

$$h_{ik}(x) = (\delta_i{}^\alpha\,\delta_k{}^4 + \delta_i{}^4\,\delta_k{}^\alpha)L_\alpha(t) + \tfrac{1}{2}K_{ik}(t)\delta_{\mu\nu}x^\mu x^\nu + O^3 \tag{2.11}$$

when expanded about the worldline C: $(0, 0, 0, t)$. Here O^3 denotes terms of third and higher orders in x, and L_α and $K_{ik} = K_{ki}$ are as yet unspecified functions of t. The h_{ik} are assumed to have the same regularity properties as g_{ik}, and to fall very rapidly (say, exponentially) to zero outside the tube $\mathscr{T}$. Note that (2.10) is *not* a coordinate transformation, but a change in the geometry of Σ, formally represented as a finite perturbation of the metric tensor in a *fixed* coordinate frame. Hence the C-gauge, which defined the frame (x) on C up to certain second metric derivatives, should be valid in the perturbed geometry as well. This is easily established: we read off from (2.11) that the only nonvanishing h_{ik} and first and second derivatives of h_{ik} on C are

$$h_{\alpha 4} \overset{C}{=} L_{\alpha}, \qquad h_{\alpha 4,4} \overset{C}{=} L_{\alpha,4}, \qquad h_{\alpha 4,44} \overset{C}{=} L_{\alpha,44}, \qquad h_{ik,\mu\nu} \overset{C}{=} K_{ik}\delta_{\mu\nu} \tag{2.12}$$

It then follows immediately that the C-gauge (2.2), (2.3), (2.5), (2.6) holds in the perturbed metric as well. Henceforth barred quantities refer to the perturbed or modified geometry $\bar{\mathscr{G}}$ in Σ, and unbarred correspondingly to $\mathscr{G}$.

Before specifying the perturbation in detail, let us consider its general effects on the matter and gravitational fields in Σ. The former may be expected to change: $\phi(x) \to \bar{\phi}(x) = \phi(x) + \Delta\phi(x)$. We shall assume that $\Delta\phi$ always falls off outside $\mathscr{T}$ like h_{ik}. This is consistent with the basic assumption that the fields generated by a test particle are not physically observable outside the near zone; thus $\Delta\phi$, like h_{ik}, is effectively localized inside $\mathscr{T}$.

This raises the question of *sources* in Σ. Namely, it follows from the local validity of the Principle of Equivalence that the matter field equations (1.1) have the same form in $\bar{\mathscr{G}}$ as in $\mathscr{G}$: they both reduce to the Lorentz-invariant equations of Special Relativity in the appropriate limit. Hence the matter energy tensor $\bar{E}_{ik}(\bar{\phi})$ has the same algebraic structure as $E_{ik}(\phi)$. Likewise, the gravitational field equations (1.2) hold in form both in $\mathscr{G}$ and $\bar{\mathscr{G}}$. However, the new Einstein tensor $\bar{G}_{ik}$ will not generally have the same algebraic properties as G_{ik}, which means that the source of $\bar{G}_{ik}$ in (1.2) cannot be $\bar{E}_{ik}(\bar{\phi})$ alone. Thus the perturbation (2.10) will in general 'create' new matter field sources, perceptible only inside the tube $\mathscr{T}$. The physical interpretation of these new sources, and the possibility of avoiding them, will be discussed in Section 4; their possible existence will not affect the conclusions of this section or the next.

The Riemann tensor on C changes by the amount

$$\begin{aligned} \Delta R_{iklm} &\equiv \bar{R}_{iklm} - R_{iklm} \\ &\overset{C}{=} \tfrac{1}{2}(h_{il,km} + h_{km,il} - h_{im,kl} - h_{kl,im}) \end{aligned} \tag{2.13}$$

Equation (2.7) for C still holds in $\bar{\mathscr{G}}$, taking the form

$$\bar{\Gamma}_{\alpha,44} \overset{C}{=} \bar{g}_{4\alpha,4} \overset{C}{=} -\bar{f}_{\alpha} \tag{2.14}$$

so that

$$\bar{f}_{\alpha} \overset{C}{=} f_{\alpha} - h_{\alpha 4,4} \overset{C}{=} f_{\alpha} - L_{\alpha,4} \tag{2.15}$$

In order to make C a geodesic in $\bar{\mathscr{G}}$, we must therefore take

$$L_{\alpha}(t) = \int_{t_0}^{t} f_{\alpha}(u)du \tag{2.16}$$

It will be convenient to take the constants of integration such that $\bar{g}_{\alpha 4} \overset{C}{=} 0$, i.e. $\bar{g}_{ik} \overset{C}{=} \eta_{ik}, \bar{g}_{ik,l} \overset{C}{=} 0$. The Einstein tensor on C then becomes

$$\bar{G}_{ik} \overset{C}{=} \eta^{ad} \bar{R}_{aikd} - \tfrac{1}{2}\eta_{ik}\eta^{ad}\eta^{bc}\bar{R}_{abcd} \tag{2.17}$$

whence, by (2.13),

$$\begin{aligned} P_{ik} &\equiv \bar{G}_{ik} - \eta^{ad} R_{aikd} + \tfrac{1}{2}\eta_{ik}\eta^{ad}\eta^{bc} R_{abcd} \\ &\overset{C}{=} \eta^{ad} \Delta R_{aikd} - \tfrac{1}{2}\eta_{ik}\eta^{ad}\eta^{bc} \Delta R_{abcd} \end{aligned} \tag{2.18}$$

This may be written in matrix form as follows:

$$P_A(t) = \sum_{B=1}^{10} M_{AB} K_B(t), \qquad A, B = 1, 2, \ldots, 10 \tag{2.19}$$

the details of ordering, etc., are set out in Appendix A. The column vector P_A is the left-hand side of (2.18) written as an ordered set of ten functions of t; they depend on both the original space-time curvature on C and on the value of the Einstein tensor on C *after* the perturbation. The 10×10-matrix M_{AB} is numerical; its components may be found by substituting (2.13) into (2.18), see Appendix A.

We may now ask: given an arbitrary unperturbed curvature R_{iklm} in Σ, how freely can we choose the $\bar{G}_{ik}$ and still be assured that (2.19) possesses solutions $K_B(t)$? In Appendix A it is shown that the matrix M_{AB} in (2.19) is nonsingular, and hence that (2.19) possesses unique solutions $K_B(t)$ for any choice of $P_A(t)$. Since the P_A depend on both the unperturbed space-time curvature R_{iklm} and the perturbed Einstein tensor $\bar{G}_{ik}$ on C, by (2.18), it follows that: for any set of background matter and gravitational fields θ in $\mathscr{G}$, the perturbation (2.10) can be chosen such as to (i) make C geodesic in $\bar{\mathscr{G}}$, and (ii) give $\bar{G}_{ik}$—and hence the matter field energy tensor $\bar{E}_{ik}$—any desired value on C. Hence the particle moves on a geodesic in $\bar{\mathscr{G}}$, regardless of the matter fields in its immediate vicinity. This we interpret as follows: the particle has been 'decoupled' from the matter fields in Σ, and is now to be regarded as 'neutral', its matter charge having been 'transferred' into the surrounding local space-time geometry. An analogous argument holds for any coupling of the particle to the inhomogeneities of the gravitational field, e.g. the spin-curvature coupling to the Riemann tensor (Papapetrou, 1951): ΔR_{iklm} on C, depending only on second spatial derivatives of h_{ik}, is left completely unrestricted by the requirement (2.16), which constrains C to be geodesic in $\bar{\mathscr{G}}$; hence $\bar{R}_{iklm}$ may be ascribed arbitrary values on C, without affecting the motion of the particle in $\bar{\mathscr{G}}$. From this we conclude that the particle may now be regarded as 'spinless'—i.e. as a mass monopole in the geometry $\bar{\mathscr{G}}$.

3. *Differential Forces; Non-Riemannian Geometry*

We now consider the phenomenon of *deviation*: i.e. the differential forces between neighbouring identically charged particles in the force field

f^k, denoting the worldlines of the particles in Σ by C and C'. If ϵX^a (ϵ = infinitesimal constant) is the separation vector between C and C' in the instantaneous 3-space t = const., the change of X^a in $\mathscr{G}$ is given by

$$\frac{D^2 X^a}{Ds^2} = \left(R^a{}_{bcd}\frac{dx^b}{ds}\frac{dx^c}{ds} + f^a{}_{;d}\right) X^d \tag{3.1}$$

where ; means covariant derivative in Riemannian geometry. It is thus produced partly by the ordinary *geodesic deviation* (Synge, 1960, p. 19) between C and C'—or, rather, between geodesics tangent to C, C' at t = const.—and partly by the differential matter force due to the inhomogeneity of the external field f^k. While the former is a purely geometric effect, independent of the intrinsic properties of the particles, the latter depends on the particle mass and charges, as well as on the geometry $\mathscr{G}$.

How, then, can we describe the total differential force (3.1) in geometric terms? First of all, we note that the metric perturbation (2.10) cannot affect the motion (3.1) at all, for the following reason: the particles C, C' must be assumed *not to interact*. Indeed, the phenomenon of deviation can only be meaningfully discussed for noninteracting particles: since the particles are formally treated as infinitesimally separated, any interaction force between them will go to infinity, and blow up the system. Physically, of course, the basic assumptions are (i) that the particle separation is so small that it can be treated to a good approximation as infinitesimal in the given external gravitational and matter fields, and (ii) that the masses and charges are then small enough, so that any interaction between the particles may be neglected. In the Infeld–Schild–Chase method, this is ensured by letting the masses and charges go to zero sufficiently fast, as the particles come closer together. In our approach, the requirement of no interaction means that the particle near zones *must not overlap*. Hence the metric perturbations (2.11) in the near zones of C and C' are not correlated in any simple way; in particular, the Riemann tensors on C and C' need *not* be infinitesimally different in $\bar{\mathscr{G}}$.

The problem is, thus, formally one of differing 'standards of infinitesimality': on the scale of lengths at which the particle separation is taken to be infinitesimal, the near zone cross-sections shrink to zero, and the finite, localized perturbation (2.11) is not well defined. Or, formulated from the other point of view: on the scale at which the near zones have finite cross-sections, C and C' are not infinitesimally separated, and the notion of differential forces between them ceases to apply.

We shall take the particle separation to be infinitesimal, in the sense defined above, and the metric tensor on the worldlines to be the unperturbed g_{ik}. We now consider the problem of geometrizing the deviation between C and C'; in particular, we ask whether it is possible to describe the relative motion (3.1) as a deviation between *geodesics* in some space. It is immediately clear that this must be more general than Riemannian space-time, since the right-hand side bracket of (3.1) will not in general have the symmetries of a Riemann tensor; in fact, these symmetries require that the force field f^k be *conservative*: $f_{[k;l]} \equiv 0$. So, we try to geometrize (3.1) in a general affine space $\mathscr{A}$, endowed with an affine connection $L^i{}_{kl}(x)$.

In Appendix B we have reviewed a few properties of affine spaces. It turns out to be sufficient for our purposes to consider a *metric* affine space $\mathscr{A}$; the physical justification for this restriction is discussed in section 4. Comparing equation (B17) for geodesic deviation in $\mathscr{A}$ with equation (3.1) for the toal deviation between trajectories of identically charged particles in Riemannian space-time $\mathscr{R}$, we identify the matter field differential force with the torsion terms, thus:

$$f_{k;l} = (U_{klm;p} - U_{ksm}\,U^s{}_{lp} - U_{kls}\,U^s{}_{mp})\,V^m\,V^p \tag{3.2}$$

with symmetric and antisymmetric parts

$$f_{(k;l)} = U_{skp}\,U^s{}_{lm}\,V^m\,V^p \tag{3.3}$$

$$f_{[k;l]} = (U_{klm;p} - U_{kls}\,U^s{}_{mp})\,V^m\,V^p \tag{3.4}$$

The change of *length* of the separation vector X^k is found, using (B.14)–(B.17), to be:

$$\frac{\mathscr{D}^2}{\mathscr{D}s^2}(X^k X_k) = 2X_k \frac{\mathscr{D}^2 X^k}{\mathscr{D}s^2} = 2X^k X^l (R_{kmpl} + U_{skp}\,U^s{}_{lm})\,V^m\,V^p \tag{3.5}$$

Thus, as expected, only the symmetrized force gradient (3.3)—i.e. the expansion and shear—contribute to the actual strains induced between the particles C, C'. The antisymmetrized force gradient (3.4) appears as a 'rotational' or 'spinlike' degree of freedom in the geometry of $\mathscr{A}$, enabling the latter to simulate the 'twisting' or 'curling' action of nonconservative forces on charged particle trajectories.

We have introduced 24 new field components—the torsion tensor $T^i{}_{kl}$—to describe 16 matter field gradients $f_{k;l}$. We may thus choose auxiliary conditions on the torsion, and it will be convenient to require that it—or, equivalently, the tensor $U^i{}_{kl}$—be *traceless*:

$$T^k{}_{lk} \equiv 0 \equiv U^k{}_{lk} \tag{3.6}$$

It is then possible to deduce that identically charged test particles have L-geodesic motions (see Appendix B), from an L-conservation law for the energy tensor of the particles in $\mathscr{A}$. For a cloud of noninteracting point particles, we may define an averaged scalar mass density μ, and the energy tensor

$$\overset{(p)}{E}{}^{ik} \equiv \mu W^i\,W^k \tag{3.7}$$

where W^k is the 4-velocity field associated with the streamlines of the mass distribution μ. We then have

$$\overset{(p)}{E}{}^{ik}{}_{|k} = (\mu W^k)_{|k} W^i + \mu W^i{}_{|k} W^k = \mu W^i{}_{|k} W^k = \mu \frac{\mathscr{D} W^i}{\mathscr{D}s} \tag{3.8}$$

by (B.18), since the mass current vector μW^k is assumed to be locally conserved (the meaning of local conservation in $\mathscr{A}$ is discussed below). Hence the L-conservation law

$$\overset{(p)}{E}{}^{ik}{}_{|k} = 0 \tag{3.9}$$

implies that the streamlines of μ are L-geodesics,

$$\frac{\mathscr{D}W^i}{\mathscr{D}s} = 0 \tag{3.10}$$

in complete analogy to the corresponding situation

$$E^{ik}{}_{;k} = 0 \Rightarrow \frac{DW^k}{Ds} = 0$$

(with Γ-derivatives) for a cloud of noninteracting neutral mass monopoles in $\mathscr{R}$. The geodesic postulate for charged particle motion in $\mathscr{A}$ then follows, similarly as in $\mathscr{R}$, upon going over to a discrete description of the particle distribution. Note also that, since the particle energy tensor is not Γ-conserved,

$$0 \neq k^i \equiv \overset{(p)}{E}{}^{ik}{}_{;k} = -U^i{}_{ab}\overset{(p)}{E}{}^{ab} \tag{3.11}$$

We may interpret (3.11) as saying that there is a net force k^i acting locally on the matter distribution μ. [This force per unit volume, which acts on a volume element containing very many particles—the idealization to a continuum description of matter—should not be confused with the force per unit mass (1.4), which acts on a single particle.] There is, however, a certain arbitrariness in the definition of the force (3.11), as will be discussed below.

The main concern of this paper has been with the action of external forces on test particles; and accordingly the geometry of $\mathscr{A}$, i.e. the metric and torsion tensors, have been treated as fixed external constraints on the particle motions. We now conclude this section with a few general remarks on the problems involved in considering possible dynamical properties of the torsion field.

One may introduce the torsion as a new physical field—or, rather, as representing additional dynamical degrees of freedom in the space-time geometry, not contained in the metric tensor. The essential task is then to describe the propagation of the torsion field, and its interactions, by means of field equations and conservation laws. Several such theories have been proposed. One example is the theory of Kibble (1961) and Sciama (1962) based on an action principle assuming two kinds of coupling between matter and geometry: energy to metric and spin to torsion; for non-spinning matter, their theory reduces to ordinary General Relativity. In the KS-theory, the matter spin-density is the source of a torsion field, which in turn induces an asymmetry in the matter energy tensor E_{ik}. This tensor is then, in turn, the source of a nonsymmetric tensor expression in the metric and torsion tensors and their derivatives, namely the affine Einstein tensor (B.13); the latter is the variational derivative of the KS free-field Lagrangian, which is the affine curvature scalar K in (B.12). However, it should be noted that K is only one of many possible candidates for a Lagrangian, and that it is not even uniquely the 'simplest' scalar constructible from the metric and torsion. One may, for instance, add to (B.12) other terms quadratic in the torsion, such as $U_{sab}U^{sab}$, which would lead to different Euler–Lagrange equations in the action principle. Hence there is considerable arbitrariness in the choice of field equations for the torsion, and no obvious way of picking out a unique set by requirements of 'simplicity'. (For instance, it is readily checked that the modification of the KS Lagrangian suggested above will yield Euler–Lagrange equations which differ from the KS ones only by additional quadratic terms in the torsion, and hence are not any 'less simple' in structure than the KS field equations.) This is in contrast to General Relativity, where the Riemannian curvature scalar R actually is the simplest Lagrangian constructible from the metric, in the sense that it is the only scalar that yields second order quasilinear field equations for the metric.

Finally, we discuss *conservation laws*; the following remarks will apply irrespectively of whether we assume the torsion to have dynamical properties. There is a certain ambiguity as to the physical meaning of conservation in $\mathscr{A}$. To see this, we recall two facts about affine spaces: (i) the difference between two connections, both defined over the same $\mathscr{A}$, is a tensor in $\mathscr{A}$; and (ii) a coordinate frame can be found in which the connection vanishes at any one given point in $\mathscr{A}$, if and only if the space is symmetric. Let us then consider, in a symmetric affine space $\mathscr{A}_{(\,)}$, a covariant conservation law: i.e. a statement that a covariant divergence, with respect to some symmetric connection defined over $\mathscr{A}_{(\,)}$, vanishes. Coordinates can then be found in which this law reduces locally to a continuity equation (a vanishing ordinary divergence), which expresses the idea of conservation in physical terms: the change with time of some quantity Z inside a small spatial volume is due to the flux of Z across the surface bounding this volume, so that Z does not have local sources or sinks. The trouble is, however, that *many* symmetric connections may be defined over the same $\mathscr{A}_{(\,)}$; in a Riemannian space $\mathscr{R}$, for instance, both $\Gamma^i{}_{kl}$ and (say) the object

$$\Delta^i{}_{kl} \equiv \Gamma^i{}_{kl} + R^i{}_{(k;l)} \tag{3.12}$$

(where R is the Ricci-tensor) are symmetric connections, and may be used in defining conservation laws, as described above. Moreover, the resulting laws are then physically different: if a quantity is Γ-conserved, it is not Δ-conserved. Denoting the frame in which a symmetric connection $L_{(\,)}$ vanishes locally as the *local rest frame* of $L_{(\,)}$, we may explain this by saying that the local rest frames of Γ and Δ do not coincide, but are in mutual acceleration; hence a Δ-conserved quantity, say, will have sources (depending on the tensor $R^i{}_{(k;l)}$) in a rest frame of Γ. In order to make the notion of conservation well defined, it is therefore necessary to choose the connection relative to which the law is to apply.

The choice of the Christoffel connection Γ in General Relativity is, essentially, based on the following assumptions:

(1) Special Relativity is the *local limit* of General Relativity: i.e. it is a valid theoretical description, when appropriate coordinates are chosen, of physical systems confined inside 4-volumes sufficiently small so that differential effects of space-time curvature on the systems can be neglected. (This is variously called the Principle of Equivalence and of Minimal Coupling; it clearly cannot be applied to systems directly coupled to curvature, such as spinning particle (Papapetrou, 1951). In order to make the Principle physically meaningful, we must assume that there exist systems to which it does apply, and that all cases where it does not can be satisfactorily accounted for by some definite system-curvature coupling.) Hence the local continuity equations, derived from covariant conservation laws as indicated above, are correctly described by this theory.

(2) The 'appropriate coordinates' referred to in (1) are the local rest frames of Γ. Hence, if Δ (or any symmetric connection in $\mathscr{R}$ other than Γ) were chosen as the basis for covariant conservation laws in $\mathscr{R}$, then the local limit of these laws (continuity equations) would in general differ from the laws of special-relativistic physics. Since the rest frames of Γ are observationally well defined in terms of space-time measurements, the observed validity of Special Relativity for a wide range of physical systems in these frames then furnishes a strong argument for taking Γ as the basic affine connection for conservation laws in General Relativity.

In a nonsymmetric affine space the situation is more complicated, since the total connection L can *not* be locally eliminated by a coordinate transformation. Hence an L-conservation law does not have a 'special relativistic' local limit, in the usual sense: the continuity equation, as obtained in the local rest frame of some symmetric part $L_{(\ )}$ of L, will in general always contain *source terms* depending on the torsion. In this case, it is no longer obvious how the symmetric connection $L_{(\ )}$, and thus the rest frames where local physical laws are to be valid, should be defined. In the case discussed above, equations (3.7)–(3.10), we have chosen $L_{(\ )} \equiv \Gamma$; however, it may be readily checked that essentially the same result (namely, that L-conservation of $\overset{(p)}{E}{}^{ik} \Rightarrow L$-geodesic fluid streamlines W^k) would be obtained by choosing $L_{(\ )}$ equal to, say,

$$\tilde{\Gamma}^i{}_{kl} \equiv L^i{}_{(kl)} = \Gamma^i{}_{kl} + U^i{}_{(kl)} \tag{3.13}$$

because of (3.6). In the local rest frame of $\tilde{\Gamma}$, we would get an expression for the force density k^i different from that of (3.11). One could argue that the choice $L_{(\ )} \equiv \Gamma$ is to be preferred, because it separates the purely metric effects encountered in General Relativity from those due to the torsion in $\mathscr{A}$, whereas the choice (3.13) for a symmetric connection would 'mix' these two kinds of effects. Also, the Christoffel symbol Γ is, in a certain sense, the 'simplest' connection which can be defined in $\mathscr{A}$: it is the only one which is constructible as a local function of the metric tensor and its *first* derivative only. This follows from the above (i), and the fact that any tensor of odd rank, defined locally as a function of the metric tensor and its derivatives, must depend on metric derivatives of at least *second* order. However, such arguments are based more on formal simplicity than on physical considerations, and are perhaps not as convincing as in the case of General Relativity.

4. *Physical Interpretation*; *Discussion*

Summarizing briefly, we have considered a general charged test particle, acted on by some external force. It has been assumed that the particle interacts with the field, and thus modifies the space-time geometry; however, the effects of this interaction are assumed to fall off, away from the trajectory, so rapidly as to be effectively confined within the near zone. We have then studied the motion of the particle in this modified geometry $\bar{\mathscr{G}}$, regarded as an *external constraint* on the trajectory; and investigated the properties $\bar{\mathscr{G}}$ must have, in order to be able to account wholly for the particle motion, if the latter is treated as a neutral mass monopole.

Our aim has been to retain as much as possible of the properties of a point particle in Special Relativity: thus we have assumed that it remains pointlike in $\bar{\mathscr{G}}$—i.e. is at each instant localized within a locally flat (infinitesimal) 3-volume—and therefore has a well-defined trajectory in this geometry. Also, we have required that the particle motion in $\bar{\mathscr{G}}$ be determined solely by the *local* space-time geometry (metric and its first derivative) on the trajectory, and that it be in fact *geodesic* in $\bar{\mathscr{G}}$. Hence the close analogy, in our treatment, between matter field forces and curvature forces: i.e. forces arising from the coupling of multipole particles to the inhomogeneities of the gravitational field (second and higher metric derivatives), such as the spin-curvature coupling of Papapetrou (1951). Both arise because the particle carries charges, coupling it to the external field; and both are geometrized by 'transferring' the charge degrees of freedom to the *local* gravitational field. This fits in quite well with the fact that a geometrized matter field actually *is* a curvature effect, i.e. depends on second and higher metric derivatives, as noted in Section 1. It also makes unnecessary any assumptions as to whether the external matter field ϕ is geometrizable or not.

Our procedure will necessarily leave many features of the modification of geometry undetermined. A complete description of $\bar{\mathscr{G}}$ in the near zone $\mathscr{T}$ would require consideration of the particle's reaction back upon the incident field, and thus make the problem essentially one of interacting fields in $\mathscr{T}$. In particular, one would have to choose a 'particle model', which can be done in different ways, even for the same field-particle system. [In the case of an accelerated electric charge, for instance, there is (a) the usual treatment of radiation reaction, where the trajectory remains well defined, but there is no geometrization of force, no modification of the near zone geometry, and no consideration of the particle fields as such; and (b) the Misner–Wheeler wormhole model of charge, as lines of force trapped in a handle (Wheeler, 1962), where the concept of a one-dimensional trajectory is lost, and the system is one of interacting gravitational and (geometrized) electromagnetic fields.] Our approach has been to leave arbitrary the particle model (except for requiring that it be pointlike), accept the charges as phenomenological internal degrees of freedom for the particle, and investigate to what extent these degrees of freedom—or, more specifically, their effect on the particle motion—can be incorporated in the local geometry $\bar{\mathscr{G}}$ *on the trajectory*. This obviously leaves a wide choice for the modified geometry elsewhere in $\mathscr{T}$.

The perturbation (2.10) will in general create new matter field sources in the near zone $\mathscr{T}$, as discussed in Section 2. This is not necessarily an objection to our procedure; we have shown that the energy tensor of such sources cannot affect the motion of the particle in $\bar{\mathscr{G}}$. In fact, from a physical point of view we might well expect new sources to appear in $\mathscr{T}$ if the particle fields, and their back-reaction on the external fields in the near zone, are taken into account; for instance, the action of an electromagnetic field on a Dirac particle (i.e., a charged spinor field) would presumably lead to spinor field sources in the near zone. Still, it would be interesting to know whether it is indeed *possible*, in general, to choose (2.10) such that no new sources appear in $\mathscr{T}$: i.e. such that $\bar{G}_{ik}$ and G_{ik} satisfy the same algebraic conditions. Expanding $\bar{G}_{ik}$ in terms of $\bar{g}_{ik} = g_{ik} + h_{ik}$, this leads to a set of coupled nonlinear second-order equations in h_{ik}:

$$S(h_{ik,ab}; h_{ik,a}; h_{ik}) = 0 \tag{4.1}$$

We then have a Dirichlet problem: namely, that of finding solutions of (4.1) which fall off arbitrarily fast away from $\mathscr{T}$, on a family of spacelike hypersurfaces. Mathematically, this problem is rather intractable (Miranda, 1969); except in very special cases, little is known about whether solutions for such a set exist, let alone about whether they admit a boundary value problem. We have not succeeded in solving this problem, even in the simple cases of an electromagnetic or neutral scalar field. On physical grounds, however, we know that solutions of (4.1) exist; they represent mappings between different configurations of external fields θ in Σ. (These solutions even have a gauge group, corresponding to representations of the same configuration θ in different coordinate frames; this can be used to normalize the h_{ik} and their first derivatives to any desired values on C.) Moreover, many different localized configurations—i.e., geons—are known. We may therefore conjecture that solutions to the Dirichlet problem of (4.1) exist, i.e. that the perturbation (2.10) of $\mathscr{G}$ does *not necessarily* introduce new sources in $\mathscr{T}$.

One consequence of the approach adopted in this paper is that the geometrization of forces on one particle and that of differential forces on two particles represent quite separate problems. As was discussed in Section 3, this is due to the basic assumption made about test particles: they are not to interact, and hence cannot have overlapping near zones. The requirement that two particles be infinitesimally separated then introduces a scale of lengths on which the perturbation (2.11) becomes infinitesimal, so that the only meaningful definition of the metric and its derivatives on the particle trajectories is the *background field*, as extrapolated in from the far zone, see Section 1. Consideration of the total deviation force (3.1) then led to the conclusion that the metric tensor by itself does not, in general, possess sufficient degrees of freedom to be able to 'simulate' the action of external differential forces on (identically charged) point particles: i.e. to describe the resulting trajectories as deviating geodesics in some Riemannian space. A more general geometry is needed, and we have restricted ourselves to considering a metric affine space $\mathscr{A}$, see Appendix B. (The reason for assuming metricity is the following: in a nonmetric space, intervals and angles change under parallel displacement; hence, the characteristic period of a freely falling atomic clock would, in general, depend on its past history. This is, as is well known, a basic physical objection against nonmetric theories, such as, e.g. Weyl's unified field theory [see, for instance, Adler *et al.*, 1965)], where the geometry is nonmetric, though torsionless.) It was found that the torsion field in $\mathscr{A}$ could be used to describe quite general differential forces on test particles.

We stress that the notion of a torsion field, as introduced here, has nothing to do with the idea of a non-Riemannian unified field theory, such as have been proposed by Einstein and others; the torsion is not a dynamical field variable, but an external constraint imposed on the motion of particles. In fact, it can be regarded simply as an auxiliary field, used to describe the differential force (3.1) in geometric terms. Nor is it in conflict with the idea of geometrodynamics: that physical phenomena should in principle be describable in terms of Riemannian geometry only. To see this, consider the Misner–Wheeler geometrization of electrodynamics (Wheeler, 1962), and take the case of an electromagnetic field acting on a pair of charged wormholes, assuming that radiation reaction and interaction between the wormholes can be neglected. In our picture, the 'particles' are treated as pointlike, and infinitesimally separated in an external (Lorentz) force field; the notion of a differential force between them then applies, and their motion is (as has been shown) geodesic in a nonsymmetric affine space. In the wormhole picture, the 'particles' are *nonlocal* objects, on the scale of lengths at which the metric and its derivatives change infinitesimally; and the notion of well-defined particle trajectories, and of differential forces between them, is no longer valid. The wormhole motion is then analogous to that of finite material bodies, coupled by their multipole moments to the curvature of *Riemannian* space-time. The difference between the two pictures is thus one of incompatible 'standards of infinitesimality', as was discussed in Section 3. Similar considerations clearly hold for any situation where both the external and particle fields are geometrizable, since (as already noted) a geometrized matter field always appears as a nonlocal feature of the Riemannian metric of space-time.

To sum up: we take the appearance of torsion to be due to the restriction we have put on our formal description of deviating test particles, namely that they be *pointlike* and *infinitesimally separated.* On a 'finer' scale of lengths, the torsion actually resolves into an averaged effect over many ripples in the space-time metric, somewhat analogous to the nonlocal geometrodynamical description of the Lorentz force (Wheeler, 1965).

In conclusion, we note that our method is only applicable to *timelike* trajectories, since it is based on the notion of a rest frame. Since massless particles with charge (spin) exist, it would be interesting to find out whether the present procedure can be extended to geometrize the forces on lightlike trajectories as well, say by the use of null tetrads on lightlike hypersurfaces (Sachs, 1964); however, the need is not perhaps very great, since the known massless particles (such as photons) are usually assumed to move along lightlike geodesics anyway.

Appendix A: Solvability of Equations (2.19)

We adopt the following scheme for ordering the ten independent components of a symmetric 4 × 4-matrix into a '10-vector':

$$\begin{array}{lcccccccccc} ik = & 11 & 22 & 33 & 12 & 23 & 31 & 14 & 24 & 34 & 44 \\ A = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \tag{A.1}$$

This is the ordering employed in the transition (2.18) → (2.19). The 10 × 10-matrix M_{AB} in (2.19) is then symmetric, and its nonvanishing components are

$$\begin{aligned} M_{12} &= M_{23} = M_{31} = \tfrac{1}{2} \\ M_{44} &= M_{55} = M_{66} = -\tfrac{1}{2} \\ M_{77} &= M_{88} = M_{99} = 1 \\ M_{1,10} &= M_{2,10} = M_{3,10} = 1 \end{aligned} \tag{A.2}$$

By standard manipulation (subtract the sum of the first three rows from the tenth row) the determinant of M_{AB} is readily found to be

$$\det(M_{AB}) = -\tfrac{3}{32} \neq 0 \tag{A.3}$$

The matrix M_{AB} is thus nonsingular; hence, the set of equations (2.19) will have a unique solution vector $K_B(t)$ for any choice of the left-hand side vector $P_A(t)$.

Appendix B: Affine Geometry

The general expressions below are adapted from Schouten (1954), with a somewhat different notation. At the end of this Appendix we give, for ease of comparison, the conventions involved in passing from Schouten's notation to ours.

A general affine space $\mathscr{A}$ is characterized by an affine connection $L^i{}_{kl}(x)$, which has the usual transformation properties, but need not have any symmetries. Its antisymmetric part $L^i{}_{[kl]} \equiv T^i{}_{kl}$, which transforms like a tensor, is called the *torsion* of $\mathscr{A}$; if $T^i{}_{kl} \equiv 0$, the connection, and the space $\mathscr{A}$, are said to be *symmetric*. The connection L defines a parallel displacement (L-displacement, for short) and a covariant derivative (L-derivative, here denoted by a stroke); e.g. for a contravariant vector

$$U^k{}_{|l} \equiv U^k{}_{,l} + L^k{}_{ml}\, U^m \tag{B.1}$$

The generalization of (B.1) to other tensors is then completely analogous to that of Riemannian tensor calculus, *except* that the order of the lower indices in L is now important; we adopt here the convention of always putting the differentiation index *last*.

One may always introduce a real symmetric nonsingular tensor field $g_{ik}(x)$ and its inverse $g^{ik}(x)$, to raise and lower indices in the usual way; however, in a general $\mathscr{A}$ it is not always possible to define such a 'metric tensor' with a vanishing L-derivative. The general expression for L in terms of g_{ik} and $T^i{}_{kl}$ is

$$L^i{}_{kl} = \Gamma^i{}_{kl} + g^{is}(T_{kls} + T_{skl} - T_{lsk}) - \tfrac{1}{2} g^{is}(g_{kl|s} + g_{ls|k} - g_{sk|l}) \tag{B.2}$$

where Γ is the Christoffel symbol of g_{ik}, constructed in the usual way. It is readily checked that Γ transforms like a connection in $\mathscr{A}$; hence, we may define a parallel displacement and covariant derivative with respect to it (Γ-displacement and Γ-derivative, the latter denoted by a semicolon), formally identical to the corresponding operations in a Riemannian space $\mathscr{R}$. The *absolute* L- and Γ-derivatives are the usual inner products of the corresponding covariant derivatives and the tangent vector $dx^k/d\lambda$ on the curve $x^k = x^k(\lambda)$; we shall denote them by $\mathscr{D}$ and D, respectively, thus e.g. for tensors U, V:

$$\frac{\mathscr{D}U^k}{\mathscr{D}\lambda} \equiv U^k{}_{|l}\frac{dx^l}{d\lambda}, \qquad \frac{DV_{kl}}{D\lambda} \equiv V_{kl;s}\frac{dx^s}{d\lambda} \tag{B.3}$$

Lengths and angles may be defined in $\mathscr{A}$ as in $\mathscr{R}$, using coordinate differences and the tensor field g_{ik} as a metric. A geodesic in $\mathscr{A}$ is any curve $x^k = x^k(\lambda)$ satisfying

$$\frac{\mathscr{D}}{\mathscr{D}\lambda}\left(\frac{dx^k}{d\lambda}\right) = 0 \tag{B.4}$$

for some choice of the curve parameter λ; those parameters for which (B.4) holds are called the *affine parameters* of the curve.

In a general $\mathscr{A}$ it is not always possible to define a metric tensor which is L-constant, i.e.

$$g_{ik|l} \equiv 0 \tag{B.5}$$

if this is possible, the space is said to be *metric*. In a metric $\mathscr{A}$, index raising and lowering commute with L-differentiation, and lengths and angles are invariant under L-displacement. The remainder of this Appendix will consider metric affine spaces only. In this case, the connection (B.2) reduces to

$$\begin{aligned} L^i{}_{kl} &= \Gamma^i{}_{kl} + T^i{}_{kl} + T_{kl}{}^i + T_{lk}{}^i \\ &\equiv \Gamma^i{}_{kl} + U^i{}_{kl} \end{aligned} \tag{B.6}$$

where it is convenient to introduce the tensor

$$U_{ikl} \equiv T_{ikl} + T_{kli} + T_{lki} \tag{B.7}$$

with the symmetry property

$$U_{ikl} \equiv U_{[ik]l} \tag{B.7a}$$

It is readily checked that the inverse of (B.7) is

$$T_{ikl} = \tfrac{1}{2}(U_{ikl} + U_{lik}) \tag{B.8}$$

Because of the symmetry properties (B.7a) and $T_{ikl} = T_{i[kl]}$, both U and T have 24 independent components.

The *curvature tensor* in $\mathscr{A}$ is in general

$$K^i{}_{klm} \equiv 2(L^i{}_{k[l,m]} + L^s{}_{k[l}L^i{}_{|s|m]}) \tag{B.9}$$

defined geometrically by L-displacing a vector around a closed infinitesimal loop, and measuring its resulting change, as in $\mathscr{R}$. Substituting (B.6) into (B.9), we find

$$K^i{}_{klm} = R^i{}_{klm} + 2(U^i{}_{k[l;m]} + U^s{}_{k[l}\,U^i{}_{|s|m]}) \tag{B.10}$$

where $R^i{}_{klm}$ is the *Riemann tensor* in $\mathscr{A}$, constructed from $\Gamma^i{}_{kl}$ in the usual way. We shall also need its contractions

$$K_{ik} \equiv K^s{}_{iks} = R_{ik} + U^s{}_{ik;s} - U^s{}_{it}\,U^t{}_{sk} \tag{B.11}$$

$$K \equiv g^{ik} K_{ik} = R + U^{sab}\,U_{sba} \tag{B.12}$$

and the 'affine Einstein tensor'

$$Q_{ik} \equiv K_{ik} - \tfrac{1}{2}g_{ik} K \tag{B.13}$$

Consider now the phenomenon of *geodesic deviation* in $\mathscr{A}$. Using (B.3), (B.4) and (B.6), it follows that a geodesic in $\mathscr{A}$ will, for any affine parameter λ, satisfy

$$\begin{aligned} 0 &= \frac{\mathscr{D}^2 x^k}{\mathscr{D}\lambda^2} = \frac{d^2 x^k}{d\lambda^2} + L^k{}_{lm}\frac{dx^l}{d\lambda}\frac{dx^m}{d\lambda} \\ &= \frac{D^2 x^k}{D\lambda^2} + U^k{}_{lm}\frac{dx^l}{d\lambda}\frac{dx^m}{d\lambda} \end{aligned} \tag{B.14}$$

Note that a geodesic with respect to the connection L (an L-geodesic) is *not* geodesic with respect to Γ. Let C and C' be two neighbouring L-geodesics with tangent vectors $V^k \equiv dx^k/d\lambda$ and separation vector ϵX^k (ϵ = infinitesimal const.) at some point λ = const. on C: i.e.

$$\frac{\mathscr{D}V^k}{\mathscr{D}\lambda} = \frac{DV^k}{D\lambda} + U^k{}_{lm}\,V^l\,V^m = 0 \tag{B.15}$$

For simplicity we may also assume (with no loss of generality) that C and C' are *initially parallel* at $\lambda = \text{const.}$, i.e.

$$\frac{\mathscr{D}X^k}{\mathscr{D}\lambda} = \frac{DX^k}{D\lambda} + U^k{}_{lm} X^l V^m = 0 \tag{B.16}$$

We then have

$$\begin{aligned}\frac{\mathscr{D}^2 X^k}{\mathscr{D}\lambda^2} &\equiv \frac{\mathscr{D}}{\mathscr{D}\lambda}\left(\frac{\mathscr{D}X^k}{\mathscr{D}\lambda}\right) = \frac{\mathscr{D}}{\mathscr{D}\lambda}\left(\frac{DX^k}{D\lambda} + U^k{}_{lm} X^l V^m\right)\\ &= \left(\frac{D}{D\lambda} + U^{\cdot}{}_{\cdot l} V^l\right)\frac{DX^k}{D\lambda} + U^k{}_{lm|p} X^l V^m V^p\\ &= \frac{D^2 X^k}{D\lambda^2} + U^k{}_{pl} V^l \frac{DX^p}{D\lambda} + U^k{}_{lm|p} X^l V^m V^p\\ &= \frac{D^2 X^k}{D\lambda^2} + (U^k{}_{lm;p} - U^s{}_{lp} U^k{}_{sm} - U^s{}_{mp} U^k{}_{ls}) X^l V^m V^p\\ &= (R^k{}_{mpl} + U^k{}_{lm;p} - U^s{}_{lp} U^k{}_{sm} - U^s{}_{mp} U^k{}_{ls}) X^l V^m V^p\end{aligned} \tag{B.17}$$

The geodesic deviation force in $\mathscr{A}$ is thus compounded of the ordinary Riemannian term and extra terms involving the torsion and its gradient.

We shall also need the following expression

$$A^k{}_{|k} = A^k{}_{;k} + U^k{}_{lk} A^l \tag{B.18}$$

connecting the L- and Γ-divergences of any vector A^k in $\mathscr{A}$.

Finally, we give the correspondence rules connecting our index conventions with those of Schouten. First of all, our affine connection is the *transpose* of Schouten's: i.e., while he puts the differentiation index *first*, we put it last:

$$L^i{}_{kl}\,(\text{Sch.}) = L^i{}_{lk}\,(\text{ours}) \tag{B.19}$$

Second, Schouten writes the affine curvature tensor thus:

$$K_{mlk}{}^{\cdot\, i}\,(\text{Sch.}) \equiv 2[\partial_{[m} L^i{}_{l]k}] + L^i{}_{[m|s} L^s{}_{l]k} \tag{B.20}$$

on transposing the connection as in (B.19), it follows that his curvature tensor is the *total transpose* of ours:

$$K_{mlk}{}^i\,(\text{Sch.}) \equiv K^i{}_{klm}\,(\text{ours}) \tag{B.21}$$

These correspondence rules make the sign convention for Schouten's Riemann tensor, see (B.10), agree with ours, as given in (2.8). It should be mentioned that Schouten's choice of kernel letters differs somewhat from ours (for instance, he writes K for the Riemann tensor and R for the curvature tensor, whereas we do the opposite); also, he uses Greek indices, while we use Latin ones. In order to display the above conventions more clearly, we have transcribed his notation to agree with ours in these respects.

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